Student ABC-123 Words= 2437 text + 128 Appendix = 2565 words

This investigation uses Newtonian gravity and centripetal motion theories plus data from nearly two-thousand earth bound satellites to determine the mass of the earth. My results are good to less than one-percent error.

**INTRODUCTION**

The ancient Greek mathematician and engineer Archimedes[[1]](#footnote-1) (of bathtub ‘Eureka’ fame) once suggested that he could move the earth, given a sufficiently large lever and a place to stand. He recognized how massive the earth was, but it was not until the 18th century when the British scientist Henry Cavendish[[2]](#footnote-2) determined theproportionality constant (referred to as big *G*) in Newton’s law of gravity. When asked what he was measuring, Cavendish replied that he was “weighing the earth.” This purpose of my investigation is to ‘weigh’ the earth, or more accurately, to determine the mass of the earth.

**METHOD**

It is amazing what Newton’s law of gravity can tell us. It can predict the motion of the planets about the sun, it can predict the tides on the earth, and it can predict the period of the moon about the earth. We learned all these things and more in my standard level class. Although the earth has only one natural satellite, the moon, there are in fact nearly two thousand artificial satellites currently circulating the earth. Actually, their orbits are elliptical; most satellites have a measurable degree of eccentricity. Nonetheless, there is a characteristic average orbit radius and orbit period, and from this and a few other known facts, my investigation will establish the experimental value for the mass of the earth.

**ASSUMPTIONS**

The assumptions in this investigation include Newton’s law of gravity and the equations of centripetal motion. I also assume reliable values of orbit altitudes, orbit periods, the average earth radius and the universal constant of gravitation, the Cavendish *G* value. None of these assume the mass of the earth.

**THEORY**

The centripetal force that keeps a satellite in circular orbit is provided by the Newtonian gravitational force.[[3]](#footnote-3) The mass of the earth is *M*, and mass of the satellite is *m*, the orbit radius (measured from the center of the earth) is *r*, the linear orbit speed is *v*, and the universal constant of gravity is *G*.



The orbit speed expressed in terms of one circular orbit path and one period *T* of time is:



Combining equations, we find:



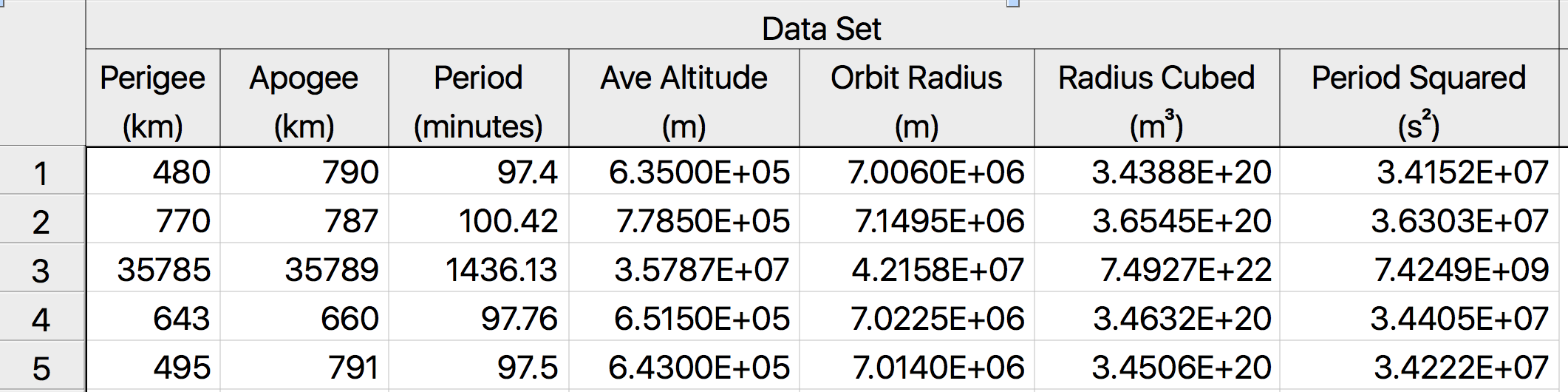
The expression[[4]](#footnote-4) of the earth’s mass *M* is:



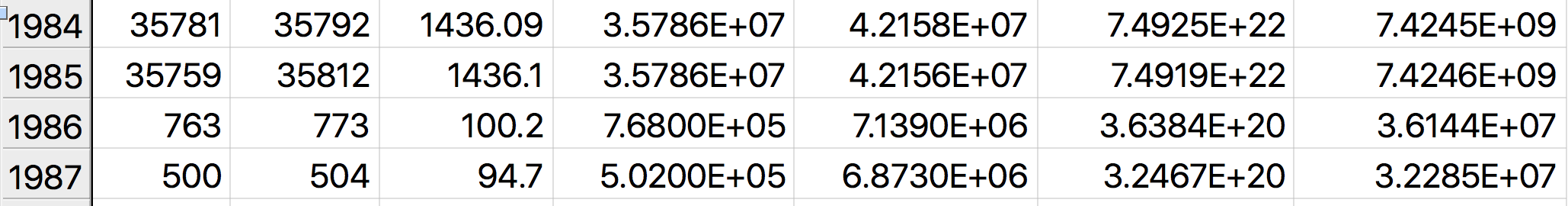
The gradient of an appropriate graph will be used along with the accepted values of the earth’s radius and the gravitational constant to solve for the mass of the earth.

**PROCEDURE**

The Union of Concerned Scientists[[5]](#footnote-5) have a listing of nearly 2000 currently orbiting satellites. The information includes: Satellite Name, Country of Origin, Ownership, Operator, User, Purpose, Orbit type, Longitude, Perigee, Apogee, Eccentricity, Inclination, Period, Launch Mass, Date, Lifetime, Launch Site, and many more details. First, I made use of data for values of perigee (km), apogee (km) and period (minutes). The *perigee* is the closest distance the satellite is to the earth’s surface, while the *apogee* is the furthest distance the satellite is from the earth’s surface. Next, I copied and pasted 1987 sets of data into my LoggerPro[[6]](#footnote-6) spreadsheet. A fourth column was made for average altitude, which was just one-half the sum of Perigee and Apogee, and then I multiplied this value by 1000 to convert kilometers to meters. The fifth column was a calculation of orbit radius, which was the sum of the accepted value of the earth’s radius and the average orbit altitude. The next column was the orbit radius cubed. The last column was a calculation of the period data in minutes converted into seconds, and then squaring the value. Several of the first and last data sets are shown below. I did not include the names of the satellites.



*Etc.*



**DATA CONSTANTS**

**Earth’s Radius.** The radius of the earth is taken from the Science World Astronomy web site by Wolfram.[[7]](#footnote-7) Uncertainties were not included.



The earth is not a perfect sphere, of course. The equatorial radius (6378 km) is greater than polar radius (6356 km)[[8]](#footnote-8) and this is a variation of about 0.3%. Technically, the earth’s shape is an ellipsoid of revolution; the shape of such an oblate spheroid is explained online at the Wolfram Mathworld[[9]](#footnote-9) web site. Therefore, there is not one absolute value of the earth radius, due to the bulging shape of the earth’s sphere and due to all the mountains. Nonetheless, if the shape of the earth were reduced to the size of a billiard ball, the model would be many times smother than a real billiard ball, or so I am told by my knowledgeable physics teacher. That is, compared to the rough and variable surface of the earth, the radius of the earth is magnitudes larger and so we can ignore the mountains and the bulging shape of the earth. Mount Everest at an altitude of 8.85 km is only about 0.14% of the earth’s radius, a value that makes no difference in the accepted value of the earth’s radius when you consider the number of significant figures. Therefore, it is justified to take the accepted value of the earth’s radius as the effective volumetric mean radius value.

**Universal Gravity Constant.** Often referred to as the Cavendish constant of universal gravitation, it is the constant required in Newton’s law of gravity. This is better known as Big *G*.The Wikipedia[[10]](#footnote-10) states an accepted value of *G* that is good to one part in 8300, so the uncertainty here is about ±0.12%.



**GRAPH FOR DATA ANALYSIS**

After processing the orbit radius and the orbit period according to theory, a graph of orbit radius cubed again period squared was made.

***Graph #1: Radius Cubed against Period Squared***



The gradient of the graph is now used to calculate the mass of the earth.





The accepted value[[11]](#footnote-11) of the mass of the earth is known to about 0.01% and is:



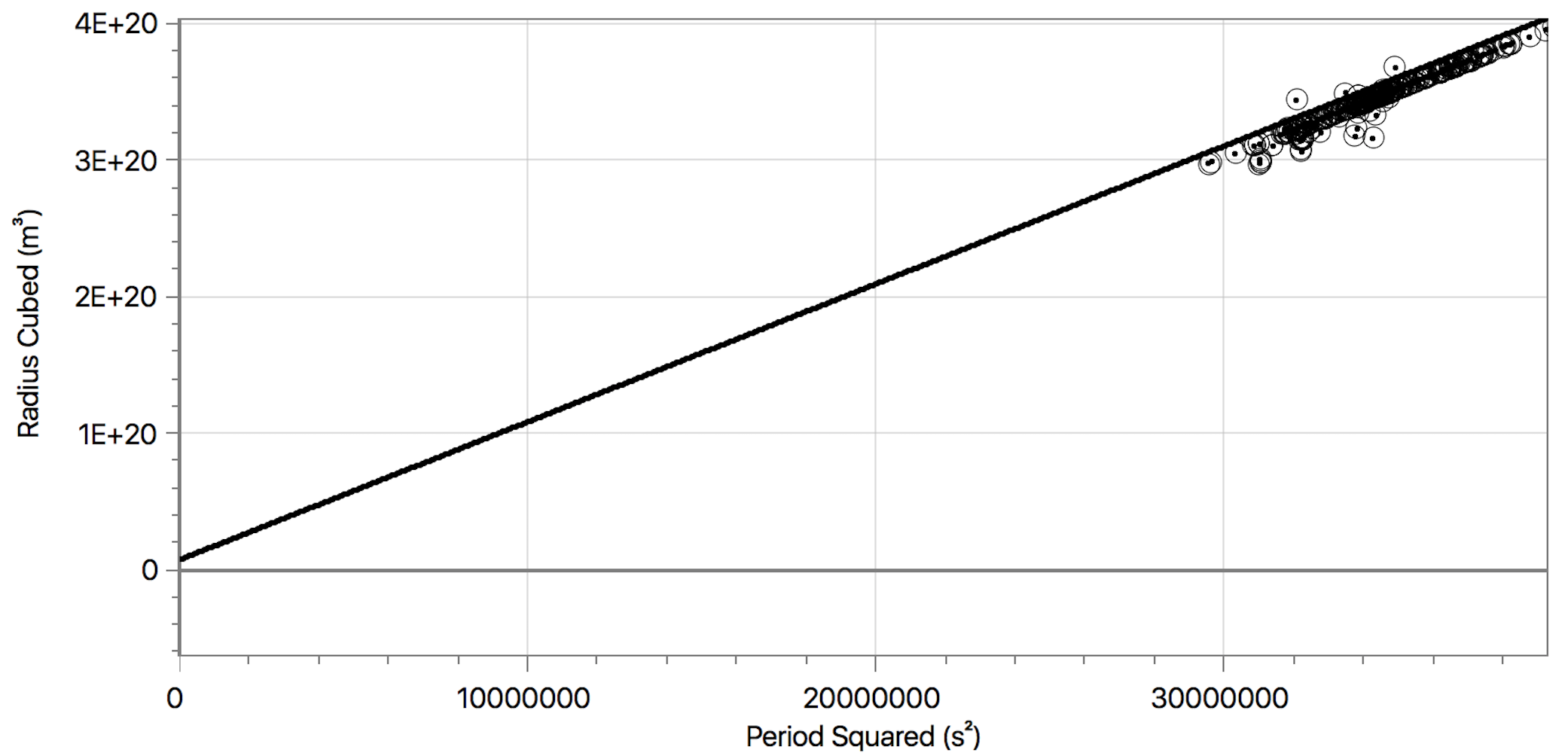
Without appreciating any experimental uncertainties, and just comparing my value to the accepted value, my experimental value is only 0.048558% or about 0.05% off.

**EVALUATION**

**Proportional Data.** According to theory, Graph #1 should be proportional. A best-fit linear line should then go through the origin; Graph #2 shows this is nearly true.

***Graph #2: Closeup of Graph #1’s Y-Axis intercept of***

***Radius Cubed against Period Squared***



The statistical value and its uncertainty for the *y*-axis intercept is:



Expressed with the same exponential power, we can write:



The statistical range is from a minimum to a maximum of:

This means that a zero-zero origin is clearly included within the statistical range. This confirms the likelihood of a proportional function.

**Linear Data.** Referring back to Graph #1, we can say that the computer’s statistical analysis yields a gradient and a standard deviation uncertainty values of:



When expressed in the same exponential power:



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The statistical variation in the gradient is slightly less than a 0.05%. We can easily accept the best-fit linear line on Graph #1 as the appropriate function.

**Random Data.** Graph #1 appears very linear and nearly no random variation; upon a closer look, however, in Graph #3 we do see some statistical variation. This graph is for low altitudes only.

***Graph #3: Closeup of a Selection of Low Altitude Satellite Data.***



There are various implicit factors responsible for the apparent random data points, such as orbit eccentricity and orbit inclination. Also, when a quantity is squared or cubed, the manipulation augments any initial variation. There are more theoretical details not considered here, as well as the method of measuring altitude and possible uncertainties. Such subtle concerns are beyond the scope of this investigation.

**Experimental Uncertainties.** The various data uncertainties were all very small. The value of *G* was ±0.012%. The earth radius values had a range of 0.3% but given the seven significant digits, and using least count, the radius value would have a minimum uncertainty of only 2 x 10–5 %. The perigee and apogee values were given in kilometers with least count to ±one kilometer. The range was from 360 km to 114,000 km. Typically 35,000 km, so the least count uncertainty ranges would be from 0.27% to 0.0009%. All period data was given in minutes and had one-tenth or one-hundredth of a minute precision. The absolute uncertainty based on least-count is then ±0.6 second and in some cases ±0.06 second. Periods ranged from 72 minutes to 3800 minutes with typical values of 1400 minutes. A plausible percentage would be 0.007% error, sometimes more, sometimes less. Finally, I checked several of the satellite’s data as found in other sources, such as Encyclopedia Astronautical,[[12]](#footnote-12) and I found that the source I used provided the most precise values.

**CONCLUSION**

Adding together all these estimated uncertainties, the total uncertainty for my exper-imental value would be about ±0.6%. If I take 0.6% of my experimental value, I get:



Hence, my value for the mass of the earth is:



Recall that my value was slightly higher than the accepted value by only 0.049%



However, taking ±0.6% uncertainty for the experimental value, and calculating the low end my result easily includes the accept value for the mass of the earth.



The accepted value of 5.97 is thus between a low of 5.94 and a high of 6.01 (x1024 kg).

Expressing the experimental uncertainty to only one significant figure, I can write my value of the earth’s mass with 0.67% or about 0.7% uncertainty as follows:



I have successfully determined the mass of the earth.

**SUMMARY**

Assuming Newton’s law of gravity and centripetal motion equations, the experimental data of nearly 2000 artificial satellites was used to confirm a linear and proportional function between orbit radius cubed and orbit period squared. Using the gradient of the appropriate graph, this value was then used to produce an acceptable experimental value for the mass of the earth, a value less than 1% off from the accepted value. Moreover, with experimental uncertainties, the accepted value was included in the range of experimental values.

**IMPROVEMENTS**

The statistical variation of the data seems to be the only limiting factor. Perhaps a mathematical appreciation of inclinations and eccentricity, and the assumption of orbital radius could be developed into a more precise mathematical model. Nonetheless, my results are less than 1% off the accepted values so it is hard to think of improvements. A more sophisticated method, however, might consider satellites with very small or no eccentricity as well as only very large altitudes, thus reducing the percentage of uncertainties in the basic data.

Although the best-fit line and its uncertainty includes a zero-zero origin on my graph, there is a slight systematic shift upwards on the radius cubed axis (which is be seen in Graph 2). This might be due to the method of averaging perigee and apogee. I used the standard arithmetic mean, where I calculated one-half the sum of the perigee and apogee values. Elliptical satellite orbits are far more complicated that I appreciate in this study.[[13]](#footnote-13)

It might be that a *geometric mean* (the square root of the product of perigee and apogee) is more appropriate. The *geometric mean* is slightly less than the *arithmetic mean*. When a mean value is added to the earth’s radius and the result is cubed, the slight reduction in the resulting orbit radius value is obtained; hence, the best-fit line would be closer to the zero origin. There is also the notion of the harmonic mean, and this might be more appropriate for Kepler’s third law.[[14]](#footnote-14) The harmonic mean value is a line perpendicular to a straight line connecting the perihelion and aphelion positions but running perpendicular to this line and originating at the center of the earth’s mass. Other complications include the oscillation of the earth’s diameter (in and out) every 20.5 minutes, although I have argued this is negligible (but a fixed diameter is one assumption). Perhaps selecting only satellites with near perfect circular orbits and only polar or equatorial paths would reduce the uncertainty.

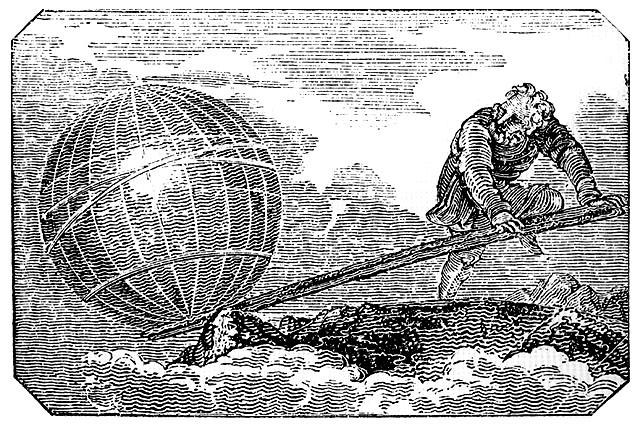
There were many other assumptions and simplifications in my method. A revised method based on much a higher precision of data would significantly improve my results. For example, NASA’s Deep Space Network can follow the precise motion of a satellites better than a factor of 1 part in 10 million when using gravitational potential.[[15]](#footnote-15) In this advanced theory, the actual orbit speeds of satellites is determined by the gravitational potential the satellites experiences.

Kepler’s law of period and orbit radius works well for solid, non-rotating and perfectly spherical bodies but the earth rotates, is not perfectly spherical, and although solid for the most part, deep inside it is not solid, also the oceans are not solids. Using NASA’s data, a far more precise calculation could be determined for the mass of the earth.

Finally, please don’t ask me about Einstein’s theory of gravity as it relates to this study.

**FURTHER STUDIES**

Further studies might use the same database and other accepted data to determine the gravitational constant *G*, the radius of the earth, or the same method could be applied to the moons of Jupiter to determine the mass of Jupiter, or (with some stretch of the imagination) perhaps data could be found to be used to determine the mass of our universe.



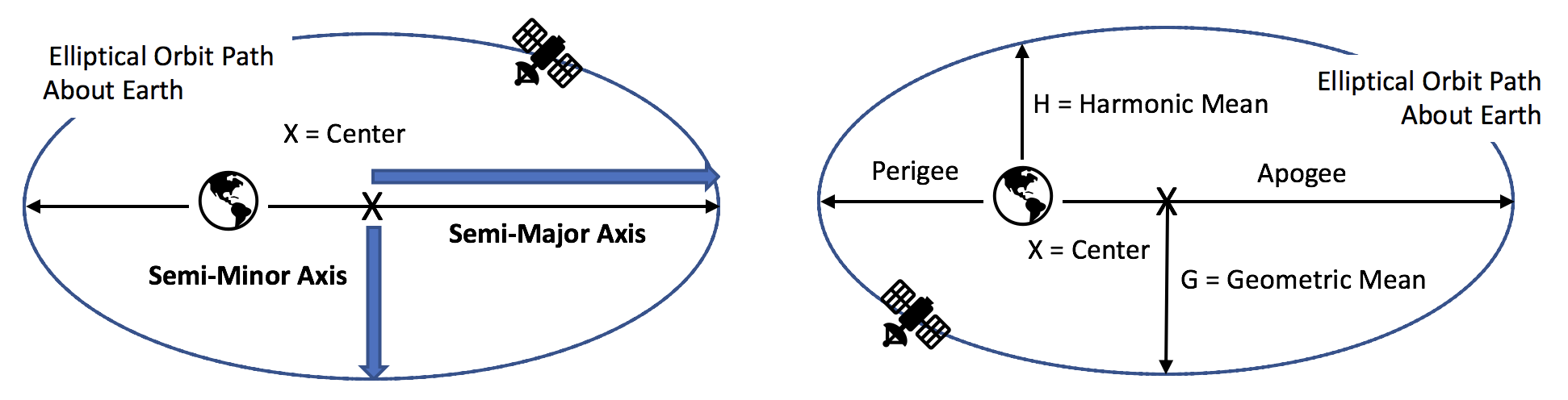
*19th-century illustration of Archimedes’ comment about*

*“give me a place to stand on, and I will move the earth.”*

Image from <https://en.wikipedia.org/wiki/Earth_mass>

**Appendix: Technical Terms[[16]](#footnote-16)**

The perigee is the shortest distance to the satellite while the apogee is the longest distance. These distances are measured from the center of the earth. The Perihelion is where the orbiting body nearest to the orbital focus; the aphelion is the point in the orbit where the body is farthest from its focus. The geometric mean G and the harmonic mean H are indicated in the follow sketches. I do not know the mathematical description of these distances.



There is a variety of orbits: low orbits, like the European space station with a period of 1.5 hours; GPD orbits with a period of 12 hours; and geosynchronous orbits with a natural period of 24 hours.

1. <https://www.cs.drexel.edu/~crorres/Archimedes/contents.html> [↑](#footnote-ref-1)
2. <https://www.britannica.com/science/gravity-physics/Interaction-between-celestial-bodies#ref210856> [↑](#footnote-ref-2)
3. See pages 19–23 and 25–35, “Gravity from the ground up” by Bernard Schutz, Cambridge University Press [↑](#footnote-ref-3)
4. The relationship between period and orbit radius is known as Kepler’s third law. The ratio of radius cubed to period squared is a constant for any given orbiting system. The actual theory concerns elliptical orbits and involves both masses, *M* and *m*, as in binary stars. For earth bound satellites, however, the mass of the satellite *m* drops out of the simplified expression. This also happens when the sun is the center of motion and the solar system planets orbit the sun. <http://hyperphysics.phy-astr.gsu.edu/hbase/kepler.html#c6> [↑](#footnote-ref-4)
5. <https://www.ucsusa.org/nuclear-weapons/space-weapons/satellite-database#.XB_3W89Kgck> [↑](#footnote-ref-5)
6. <https://www.vernier.com/products/software/lp/> [↑](#footnote-ref-6)
7. <http://scienceworld.wolfram.com/astronomy/EarthRadius.html> [↑](#footnote-ref-7)
8. NASA fact sheet*:* <https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html> [↑](#footnote-ref-8)
9. <http://mathworld.wolfram.com/OblateSpheroid.html> [↑](#footnote-ref-9)
10. <http://units.wikia.com/wiki/Gravitational_constant>

    Also see the National Institute of Standards and Technology for a value of *G* and its standard error. <http://physics.nist.gov/cuu/index.html> [↑](#footnote-ref-10)
11. <https://en.wikipedia.org/wiki/Earth_mass>

    Also see NASA: <http://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html> [↑](#footnote-ref-11)
12. See for example, <http://www.astronautix.com/index.html>

    Also see Gunter’s Space Page: <https://space.skyrocket.de/index.html> [↑](#footnote-ref-12)
13. Kepler’s law uses the semi-major axis as the orbital distance. The advanced mathematics of this

    harmonic mean is beyond the scope of my study, but see the references:

    <http://hyperphysics.phy-astr.gsu.edu/hbase/kepler.html>

    <https://www.radio-electronics.com/info/satellite/satellite-orbits/satellites-orbit-definitions.php>

    <http://www.polaris.iastate.edu/EveningStar/Unit4/unit4_sub3.htm> [↑](#footnote-ref-13)
14. See the technical article:

    “Don’t demean the geometric mean” by Sanjoy MahajanAm. J. Phys. 87 (1), January 2019, pages 75-77 [↑](#footnote-ref-14)
15. “Juno at Jupiter” by David Stevenson in *Physics Today****,*** September 2020, Vol. 75, No. 9, pages 62+. [↑](#footnote-ref-15)
16. See the following online resources.

    <https://ipfs.io/ipfs/QmXoypizjW3WknFiJnKLwHCnL72vedxjQkDDP1mXWo6uco/wiki/Kepler's_laws_of_planetary_motion.html>

    <https://en.wikipedia.org/wiki/Semi-major_and_semi-minor_axes>

    <https://www.bing.com/images/search?q=semi+major+axis+or+orbit&qpvt=semi+major+axis+or+orbit&FORM=IGRE>

    —end— 06 August 2024 [↑](#footnote-ref-16)